

DIMENSION OF SPACES OF POLYNOMIALS ON ABELIAN TOPOLOGICAL SEMIGROUPS

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ABSTRACT. In this paper we study (continuous) polynomials $p : J \rightarrow X$, where J is an abelian topological semigroup and X is a topological vector space. If J is a subsemigroup with non-empty interior of a locally compact abelian group G and $G = J - J$, then every polynomial p on J extends uniquely to a polynomial on G . It is of particular interest to know when the spaces $P^n(J, X)$ of polynomials of order at most n are finite dimensional. For example we show that for some semigroups the subspace $P_R^n(J, \mathbf{C})$ of Riss polynomials (those generated by a finite number of homomorphisms $\alpha : J \rightarrow \mathbf{R}$) is properly contained in $P^n(G, \mathbf{C})$. However, if $P^1(J, \mathbf{C})$ is finite dimensional then $P_R^n(J, \mathbf{C}) = P^n(J, \mathbf{C})$. Finally we exhibit a large family of groups for which $P^n(G, \mathbf{C})$ is finite dimensional.

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0. Introduction and notation

Throughout this paper G will denote a locally compact abelian group with dual (character) group \hat{G} . In developing a Laplace transform for such groups, Mackey[8] described the relationship between the continuous homomorphisms $\alpha : G \rightarrow \mathbf{R}$ and the continuous one parameter subgroups $\beta : \mathbf{R} \rightarrow \hat{G}$. The latter are precisely the duals $\hat{\alpha}$ of the former, defined by $\hat{\alpha}(u)(t) = e^{i u \alpha(t)}$ for $u \in \mathbf{R}$ and $t \in G$. Hence there is a non-trivial (unbounded) homomorphism $\alpha : G \rightarrow \mathbf{R}$ if and only if there is a non-trivial one parameter subgroup $\beta : \mathbf{R} \rightarrow \hat{G}$.

Riss [11] introduced a space, say $P_R(G)$, of complex-valued polynomials on G . It is the unital subalgebra generated by the homomorphisms $\alpha : G \rightarrow \mathbf{R}$ in the algebra $C(G)$ of continuous complex-valued functions (in this paper homomorphisms are always continuous). Subsequently Domar[4] defined a different space of polynomials ($P(G)$ below) and mentioned that the connection between the two classes is not obvious and a study of this problem would involve extensive structural considerations. See [4, end of Chapter II]. To the best of our knowledge this problem has not yet been resolved and this note is devoted in part to clarifying this connection.

Throughout X will denote a topological vector space over \mathbf{C} and J an abelian topological semigroup. Replacing G by J in the definition of Riss polynomials we obtain the space $P_R(J)$. Following [4] (see also [1]) a function $p \in C(J, X)$ is a *polynomial* of degree n if $p(s + mt)$ is a polynomial in $m \in \mathbf{Z}_+$ of degree at most n for all $s, t \in J$ and of degree n for some $s_0, t_0 \in J$. Such n is the degree of p ($\deg(p)$). The space of all polynomials of degree at most n is denoted $P^n(J, X)$. Moreover, $P(J, X) = \cup_{n=1}^{\infty} P^n(J, X)$, $P^n(J) = P^n(J, \mathbf{C})$ and $P(J) = P(J, \mathbf{C})$.

Basit and Pryde [1] gave other characterizations, on semigroups, of the Domar polynomials using difference operators $\Delta_h p(t) = p(t+h) - p(t)$. In particular, $p : J \rightarrow \mathbf{C}$ is a Domar polynomial of degree n if and only if all differences $\Delta_h^{n+1} p$ of order $n+1$ vanish but $\Delta_{h_0}^n p(t_0) \neq 0$ for some $h_0, t_0 \in J$.

These polynomials were then used to study unbounded measurable functions $\phi : G \rightarrow X$ with finite Beurling spectra $sp_w(\phi) \subset \widehat{G}$ (see [1], [5, p.988 for the case $w = 1, G = \mathbf{R}$]). Indeed, if $\phi \in L_w^\infty(G, X) = wL^\infty(G, X)$ for an appropriate weight w and Haar measure, then $sp_w(\phi) = \{\gamma_j : 1 \leq j \leq n\}$ if and only if $\phi = \sum_{j=1}^n p_j \gamma_j$ for some polynomials $p_j \in P_w^n(G, X) = P^n(G, X) \cap L_w^\infty(G, X)$ (see [2], [9, Proposition 0.5, p. 22]). In a similar way, in Corollary 5.3 of [2], the eigenfunctions of certain X -valued convolution operators are characterized in terms of X -valued polynomials in $P_w^n(G, X)$. For further applications of polynomials, see [3].

It is therefore natural to continue this study of polynomials on semigroups. It is of particular interest to know when the space $P^n(J)$ is finite dimensional. For example using the theory of symmetric matrices we show that for some groups $P_R(G)$ is properly contained in $P(G)$ (see Example 1.4), but if $P^1(G)$ is finite dimensional then $P_R(G) = P(G)$ (see Theorem 2.6 (c)). This addresses the above question of Domar. In section 3, we exhibit a class containing the compactly generated abelian groups for which $P^n(G)$ is finite dimensional for all $n \in \mathbf{Z}_+$.

1. Extension of polynomials

Throughout this section J is a subsemigroup of G with non-empty interior J° and $G = J - J$. It is clear that every homomorphism $\alpha : J \rightarrow \mathbf{R}$ extends uniquely to a homomorphism $\alpha : G \rightarrow \mathbf{R}$. This implies that each Riss polynomial $p : J \rightarrow \mathbf{C}$ extends uniquely to a Riss polynomial $p : G \rightarrow \mathbf{C}$. In Example 1.4, we show that $P_R^n(J)$ may be properly contained in $P^n(J)$. Nevertheless, as we prove in Theorem 1.5, every polynomial $p : J \rightarrow X$ has a unique extension to a polynomial $p : G \rightarrow X$. This last result may be of independent interest.

We recall some identities needed in the sequel. Let $\phi \in C(J, X)$, $t, s \in J$, $m \in \mathbf{Z}_+$.

$$(1.1) \quad \phi(t + ms) = \sum_{j=0}^m \binom{m}{j} \Delta_s^j \phi(t).$$

$$(1.2) \quad \Delta_s^m \phi(t) = \sum_{j=0}^m (-1)^j \binom{m}{j} \phi(t + js).$$

Moreover, if $\phi \in P^m(G, X)$ and $t, s \in G$ then

$$(1.3) \quad \phi(t - s) = \sum_{j=0}^m (-1)^j \Delta_s^j \phi(t).$$

Identities (1.1), (1.2) can be verified by induction and for (1.3) set $z = t - s$. Then $\Delta_s^{m+1} \phi(z) = 0$ and so from (1.2), $\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \phi(z + js) = 0$. Thus by (1.1) $\phi(t - s) = \sum_{j=1}^{m+1} (-1)^{j+1} \binom{m+1}{j} \phi(t + (j-1)s) = \sum_{j=1}^{m+1} (-1)^{j+1} \binom{m+1}{j} \sum_{k=0}^{j-1} \binom{j-1}{k} \Delta_s^k \phi(t) = \sum_{k=0}^m (\sum_{i=k}^m (-1)^i \binom{m+1}{i+1} \binom{i}{k}) \Delta_s^k \phi(t) = \sum_{j=0}^m (-1)^j \Delta_s^j \phi(t)$, because $\sum_{i=k}^m (-1)^i \binom{m+1}{i+1} \binom{i}{k} = \sum_{i=0}^m (-1)^i \binom{m+1}{i+1} \frac{1}{k!} D^k t^i \Big|_{t=1} = \frac{1}{k!} D^k (t^{-1} \sum_{j=1}^{m+1} (-1)^{j+1} \binom{m+1}{j} t^j) \Big|_{t=1} = \frac{1}{k!} D^k (t^{-1} (1 - (1-t)^{m+1})) \Big|_{t=1} = (-1)^k$.

Examples 1.1.

(a) If G is \mathbf{Z} or \mathbf{R} then $P^n(J, X)$ is the space of ordinary polynomials. Indeed, that each (ordinary) polynomial is in $P^n(G, X)$ is clear. Conversely, if $p \in P^n(\mathbf{Z}, X)$ then

$\Delta_1^n p(m) = c$, a constant, and $\Delta_t^n(p(m) - cm^n/n!) = 0$ for all $t \in \mathbf{Z}$. An induction argument shows p is an (ordinary) polynomial. If $p \in P^n(\mathbf{R}, X)$ then $p|_{\mathbf{Z}} \in P^n(\mathbf{Z}, X)$ and so $p|_{\mathbf{Z}} = q|_{\mathbf{Z}}$ for some (ordinary) polynomial $q : \mathbf{R} \rightarrow X$. If $t = a/b$ where a, b are non-zero integers, then $p(t + m/b) = q(t + m/b)$ for $m \in -a + b\mathbf{Z}$. These are polynomials in $m \in \mathbf{Z}_+$ and so agree for all $m \in \mathbf{Z}_+$. In particular $p(t) = q(t)$ for all rationals t and, by continuity, for all reals t .

(b) If $\alpha : G \rightarrow \mathbf{C}$ is a homomorphism, then $\alpha \in P^1(G)$. Conversely, if $p \in P^1(G)$, then $\alpha = p - p(0)$ is a homomorphism.

(c) If $p \in P_w(G, X)$ and $f \in L_w^1(G)$, then $p * f \in P_w(G, X)$. Indeed $\Delta_t^{n+1}(p * f) = (\Delta_t^{n+1}p) * f$ for all $t \in G$.

Occasionally it will be necessary for us to assume that $P^n(J)$ is finite dimensional. That this condition is not always satisfied is shown by the following examples.

Example 1.2. (a) Let $G = \{s : \mathbf{N} \rightarrow \mathbf{Z} ; s \text{ has finite support}\}$ with the discrete topology. So G is countable, locally compact, σ -compact and not finitely generated. Moreover the evaluation maps $p_n : G \rightarrow \mathbf{C}$ defined by $p_n(s) = s(n)$ for $n \in \mathbf{N}$ are polynomials of degree 1 and so $\dim P^1(G) = \infty$. Note also that $J = \{s \in G : s(n) = 0 \text{ for } n < 0\}$ is a subsemigroup with $G = J - J$.

(b) The space l^2 of square summable real sequences with its norm topology is not locally compact. However, if $G = l_d^2$ denotes l^2 with the discrete topology, then G is locally compact. If $g = (g_k)$ is a bounded real sequence and $s = (s_k) \in l^2$ then

$$(1.4) \quad p(s) = \sum_{k=1}^{\infty} g_k s_k^2,$$

is a polynomial of degree 2 on both G and l^2 . Thus $P^2(G)$ and $P^2(l^2)$ are both infinite dimensional.

Lemma 1.3. Every real valued Riss polynomial which is homogeneous of order 2 can be written in the form $p = \sum_{j=1}^m \varepsilon_j \alpha_j^2$ where $\alpha_j : G \rightarrow \mathbf{R}$ are linearly independent homomorphisms and $\varepsilon_j = \pm 1$.

Proof. A scaling argument shows that $p = \sum_{i,j=1}^n c_{i,j} p_i p_j$ where p_j are linearly independent homomorphisms and $c_{i,j}$ are reals. We may also choose $c_{i,j} = c_{j,i}$. By Sylvester's law of inertia (see [7, p. 223]), matrix $(c_{i,j})$ is congruent to a diagonal matrix $\text{diag}(\varepsilon_1, \dots, \varepsilon_n)$ where $\varepsilon_j = \pm 1$ for $1 \leq j \leq m$ and $\varepsilon_j = 0$ for $m+1 \leq j \leq n$, via a non-singular matrix. Hence $p = \sum_{j=1}^m \varepsilon_j \alpha_j^2$ where the α_j are linearly independent linear combinations of the p_i . \square

Example 1.4. Assume $G = l_d^2$, $g = (g_k)$ is a bounded real sequence, $g_k > 0$ for all k and p is defined by (1.4). Then p is not a Riss polynomial.

Proof. Suppose on the contrary that p is a Riss polynomial. Then $p = \sum_{j=1}^m \varepsilon_j \alpha_j^2$ as in Lemma 1.3. The α_j are homomorphisms and therefore \mathbf{Q} -linear. We show that they are \mathbf{R} -linear. Note first that $p : l^2 \rightarrow \mathbf{R}$ is continuous. Moreover, $\Delta_h p(t) = p(h) + \sum_{j=1}^m 2\varepsilon_j \alpha_j(h) \alpha_j(t)$. Let $a = (\alpha_1, \dots, \alpha_m) : l^2 \rightarrow \mathbf{R}^m$. Since the α_j are linearly independent we can find $h_i \in l^2$, $1 \leq i \leq m$ such that $a(h_1), \dots, a(h_m)$ are linearly independent in \mathbf{R}^m . Hence the linear system $\sum_{j=1}^m 2\varepsilon_j \alpha_j(h_i) \alpha_j(t) = \Delta_{h_i} p(t) - p(h_i)$ can be solved uniquely to express the α_j as linear combinations of the $\Delta_{h_i} p - p(h_i)$. Hence each $\alpha_j : l^2 \rightarrow \mathbf{R}$ is

continuous and since it is \mathbf{Q} -linear it is \mathbf{R} -linear. Hence $a : l^2 \rightarrow \mathbf{R}^m$ has infinite dimensional kernel. This contradicts the fact that p has only one zero. \square

Theorem 1.5. Assume that J is a subsemigroup of G with non-empty interior such that $G = J - J$. Then for each n the restriction map $r : P^n(G, X) \rightarrow P^n(J, X)$ is a linear bijection. In particular, every polynomial on J has a unique extension to G .

Proof. Firstly, let $p \in P^n(G, X)$ be zero on J . For any $t \in G$ there are $u, v \in J$ with $t = u - v$. Now $p(t + mv) = p(u + (m-1)v)$ is a polynomial in $m \in \mathbf{Z}_+$ which is zero for all $m \geq 1$. It is therefore zero for $m = 0$, showing $p(t) = 0$ and r is one-to-one. Secondly, let $q \in P^n(J, X)$. We define an extension p of q to G as follows. If $t \in G$ then $t = u - v$ for some $u, v \in J$ and by (1.3) it is natural to set $p(t) = \sum_{j=0}^n (-1)^j \Delta_v^j q(u)$. If also $t = \tilde{u} - \tilde{v}$ where $\tilde{u}, \tilde{v} \in J$ we must show

$$(1.5) \quad \sum_{j=0}^n (-1)^j \Delta_v^j q(u) = \sum_{j=0}^n (-1)^j \Delta_{\tilde{v}}^j q(\tilde{u}).$$

To do this define a function $L : P^n(J, X) \rightarrow X$ by $L(q) = q(u) - \Delta_v q(\tilde{u}) = q(\tilde{u}) - \Delta_{\tilde{v}} q(u)$.

We prove by induction on $k \in \mathbf{Z}_+$ that

$$(1.6) \quad \begin{aligned} \sum_{j=0}^n (-1)^j \Delta_v^j q(u) &= \sum_{j=0}^{k-1} L(\Delta_v^j \Delta_{\tilde{v}}^j q) + \Delta_v^k \Delta_{\tilde{v}}^k q(w) \quad \text{if } q \text{ is of degree } n = 2k, \\ \sum_{j=0}^n (-1)^j \Delta_v^j q(u) &= \sum_{j=0}^k L(\Delta_v^j \Delta_{\tilde{v}}^j q) \quad \text{if } q \text{ is of degree } n = 2k + 1 \end{aligned}$$

where w is an arbitrary element of J . When $n = 0$, $q(u) = q(w)$ and when $n = 1$, $\Delta_v q(u) = \Delta_v q(w)$ and so $q(u) - \Delta_v q(u) = q(u) - \Delta_v q(\tilde{u}) = L(q)$. Hence, (1.6) holds for $k = 0$. Assume (1.6) holds for polynomials of degree less than $2k$. If $n = 2k$, we can apply (1.6) to $\Delta_v q$ and obtain

$$\begin{aligned} \sum_{j=0}^n (-1)^j \Delta_v^j q(u) &= q(u) - \sum_{j=0}^{k-1} L(\Delta_v^j \Delta_{\tilde{v}}^j \Delta_v q) \\ &= q(u) + \sum_{j=0}^{k-1} \Delta_{\tilde{v}} \Delta_v^j \Delta_{\tilde{v}}^j \Delta_v q(u) - \sum_{j=0}^{k-1} \Delta_v^j \Delta_{\tilde{v}}^j \Delta_v q(\tilde{u}) \\ &= \sum_{j=0}^k \Delta_v^j \Delta_{\tilde{v}}^j q(u) - \sum_{j=0}^{k-1} \Delta_v^j \Delta_{\tilde{v}}^j \Delta_v q(\tilde{u}) = \sum_{j=0}^{k-1} L(\Delta_v^j \Delta_{\tilde{v}}^j q) + \Delta_v^k \Delta_{\tilde{v}}^k q(u). \end{aligned}$$

If $n = 2k + 1$, we can apply this last result to $\Delta_v q$ and obtain

$$\begin{aligned} \sum_{j=0}^n (-1)^j \Delta_v^j q(u) &= q(u) - \sum_{j=0}^{k-1} L(\Delta_v^j \Delta_{\tilde{v}}^j \Delta_v q) - \Delta_v^k \Delta_{\tilde{v}}^k \Delta_v q(w) \\ &= q(u) + \sum_{j=0}^{k-1} \Delta_{\tilde{v}} \Delta_v^j \Delta_{\tilde{v}}^j \Delta_v q(u) - \sum_{j=0}^{k-1} \Delta_v^j \Delta_{\tilde{v}}^j \Delta_v q(\tilde{u}) - \Delta_v^k \Delta_{\tilde{v}}^k \Delta_v q(\tilde{u}) \\ &= \sum_{j=0}^k \Delta_v^j \Delta_{\tilde{v}}^j q(u) - \sum_{j=0}^k \Delta_v^j \Delta_{\tilde{v}}^j \Delta_v q(\tilde{u}) = \sum_{j=0}^k L(\Delta_v^j \Delta_{\tilde{v}}^j q). \end{aligned}$$

Hence, (1.6) is proved and (1.5) follows, which means p is well-defined. Note that if $t \in J$ then we can take $t = u$, $v = 0$ and so $p(t) = q(t)$. So p is an extension of q . Next, we show p is continuous. Since J has an interior point s_0 , there is an open neighborhood W of 0 in G such that $s_0 + W \subset J$. Moreover, if $t = u - v$, where $u, v \in J$ then $t = \tilde{u} - \tilde{v}$, where $\tilde{u} = s_0 + u$, $\tilde{v} = s_0 + v$. Let (t_α) be a net in G converging to t . We may suppose $t_\alpha = t + w_\alpha$ where $w_\alpha \in W$. Setting $u_\alpha = \tilde{u} + w_\alpha$ and $v_\alpha = \tilde{v}$ we find $u_\alpha, v_\alpha \in J$, $t_\alpha = u_\alpha - v_\alpha$ and $(u_\alpha) \rightarrow \tilde{u}$. So $p(t_\alpha) \rightarrow p(t)$. Finally, if $t_j = u_j - v_j$ where $u_j, v_j \in J$ and $m \in \mathbf{Z}_+$ then $p(t_1 + mt_2) = \sum_{j=0}^n (-1)^j \Delta_{v_1 + mv_2}^j q(u_1 + mu_2)$ which is a polynomial in m of degree at most n . So $p \in P^n(G, X)$, proving r is onto. \square

2. Sufficient conditions for all polynomials to be Riss

In this section we develop a method enabling us to determine the dimension of the space of polynomials $P^n(J)$ and to give conditions under which $P^n(J) = P_R^n(J)$.

Lemma 2.1. Assume that $\phi \in P^n(J, X)$ so $\phi(t + ms) = \sum_{j=0}^n a_j(t, s) m^j$ for all $t, s \in J$, $m \in \mathbf{Z}_+$. Then $a_n(t, s) = a_n(0, s)$.

Proof. Applying (1.2) to the polynomial on \mathbf{R} given by $p_k(u) = u^k$ where $0 \leq k \leq n$ and using $\Delta_1^n p_k(0) = n! \delta_{k,n}$ we get

$$(2.1) \quad \sum_{j=0}^n (-1)^j \binom{n}{j} j^k = n! \delta_{k,n} \text{ for } 0 \leq k \leq n \text{ } (0^0 = 1).$$

Now consider $\phi \in P^n(J, X)$. We have $\Delta_s^n \phi(t)$ is independent of t . By (1.2) again

$$\begin{aligned} \Delta_s^n \phi(t) &= \sum_{j=0}^n (-1)^j \binom{n}{j} \phi(t + js) = \sum_{j=0}^n (-1)^j \binom{n}{j} \sum_{k=0}^n a_k(t, s) j^k \\ &= \sum_{k=0}^n a_k(t, s) \sum_{j=0}^n (-1)^j \binom{n}{j} j^k = \sum_{k=0}^n a_k(t, s) n! \delta_{k,n} = n! a_n(t, s). \end{aligned}$$

So $a_n(t, s) = a_n(t, 0)$. □

Lemma 2.2. If $\phi \in P^n(J, X)$ and $\phi(ms) = m^k \phi(s)$ for some $0 \leq k < n$ and all $m \in \mathbf{Z}_+$, $s \in J$ then $\phi \in P^k(J, X)$.

Proof. From (1.1), we have $\phi(t + ms) = \sum_{j=0}^n a_j(t, s) m^j$, so $\phi(ms) = \sum_{j=0}^n a_j(0, s) m^j = m^k \sum_{j=0}^n a_j(0, s)$. It follows that $a_j(0, s) = 0$ for all $j \neq k$. By Lemma 2.1, $a_n(t, s) = a_n(0, s) = 0$. It follows $\phi \in P^{n-1}(J, X)$. Applying this argument repeatedly we obtain $\phi \in P^k(J, X)$. □

Proposition 2.3. If $\phi \in P^n(J, X)$ then $\phi = \sum_{j=0}^n a_j$ where $a_j \in P^j(J, X)$ and $a_j(mt) = m^j a_j(t)$ for all $m \in \mathbf{Z}_+$, $t \in J$.

Proof. From (1.1), we have $\phi(mt) = \sum_{j=0}^m \binom{m}{j} \Delta_t^j \phi(0)$ for all $m \in \mathbf{Z}_+$. But $\Delta_t^j \phi = 0$ if $j \geq n+1$ and so $\phi(mt) = \sum_{j=0}^n \binom{m}{j} \Delta_t^j \phi(0) = \sum_{j=0}^n a_j(t) m^j$, where $a_j(t) = \sum_{k=0}^n a_{jk} \Delta_t^k \phi(0)$ for some $a_{jk} \in \mathbf{R}$. Thus $a_j \in P^n(J, X)$. Moreover, $\phi(mlt) = \sum_{j=0}^n a_j(lt) m^j$. So, $a_j(lt) = l^j a_j(t)$ for all $l \in \mathbf{Z}_+$. By Lemma 2.2, $a_j \in P^j(J, X)$. □

Recall that a group is called a *torsion group* if every element has finite order. If the orders of the elements are bounded the group is said to be of *bounded order*. A group is *torsion free* if no element other than the identity is of finite order (see [6, Definition, p. 88]).

Lemma 2.4. Let K be a compact or torsion abelian topological group. Then $P(K, X) = P^0(K, X)$.

Proof. Let $p \in P(K, X)$. If K is compact, $p(K)$ is compact and if K is torsion, then the sequence $(p(t + ms))$, $m \in \mathbf{Z}_+$ is compact for each $t, s \in K$. If p is not constant then for some $t, s \in K$ one has $p(t + ms)$ is an unbounded polynomial in $m \in \mathbf{Z}_+$. This is a contradiction which proves that $\deg(p) = 0$. □

We give the proof of the following needed result stated without a proof in [8].

Theorem 2.5 (Mackey). The following statements are equivalent.

- (a) For each $t \in G \setminus \{0\}$ there is a homomorphism $\alpha : G \rightarrow \mathbf{R}$ such that $\alpha(t) \neq 0$.
- (b) $G = \mathbf{R}^m \times F$ for some $m \in \mathbf{Z}_+$ and some discrete torsion free group F .

Proof. (a) \Rightarrow (b): By the principal structure theorem [12, Theorem 2.4.1], \widehat{G} has an open and closed subgroup $\Gamma_0 = \mathbf{R}^m \times K$, where K is compact. The annihilator $H = \Gamma_0^\perp$ is isomorphic to the dual of the discrete group \widehat{G}/Γ_0 (see [12, Theorem 2.1.2, p. 35]). So H is compact and thus every homomorphism $\alpha : G \rightarrow \mathbf{R}$ is zero on H (see also Lemma 2.4). By (a) we conclude $H = \{0\}$ and therefore $\widehat{G} = \Gamma_0 = \mathbf{R}^m \times K$. Hence $G = \mathbf{R}^m \times F$ where $F = \widehat{K}$ is discrete. By (a) we also conclude that G and therefore F are torsion free.

(b) \Rightarrow (a): If $t \in F \setminus \{0\}$ then using Zorn's lemma we may establish the existence of a homomorphism $\alpha : F \rightarrow \mathbf{R}$ with $\alpha(t) \neq 0$ (see also [12, Theorem (Kaplansky), p. 44]). Then α extends readily to a homomorphism $\alpha : G \rightarrow \mathbf{R}$. Now (a) follows. \square

In the sequel for $Y = X$ or \mathbf{R} we use

$$H_0(Y) = \{x \in G : \alpha(t) = 0 \text{ for all homomorphisms } \alpha : G \rightarrow Y\} \text{ and } H_0 = H_0(\mathbf{R}).$$

Clearly $H_0(X) \subset H_0$. We claim that if $X \neq \{0\}$, then $H_0(X) = H_0$. Indeed this is clear if X is locally convex, by the Hahn-Banach theorem. Assume X is not locally convex and there is $t \in H_0$ such that $t \notin H_0(X)$. Then there is a homomorphism $\alpha : G \rightarrow X$ such that $y = \alpha(t) \neq 0$. Let $X_y = \mathbf{C}y$ and $P_y : X \rightarrow X_y$ be a projection map. Then $\beta = P_y \circ \alpha : G \rightarrow X_y$ is a homomorphism with $\beta(t) = \alpha(t) \neq 0$. But X_y is locally convex. This implies $\alpha(t) = 0$, a contradiction which proves our claim.

Proposition 2.6. If $\phi \in P^n(G, X)$, then $\phi(t) = \phi(0)$ for all $t \in H_0$.

Proof. If $n = 1$, $\phi(t) - \phi(0) = \alpha(t)$, $\alpha : G \rightarrow X$ a homomorphism. So $\phi(t) = \phi(0)$ on H_0 for all $\phi \in P^1(G, X)$. As induction hypothesis assume $\phi(x) = \phi(0)$ on H for all $\phi \in P^{n-1}(G, X)$. By Proposition 2.3, it suffices to prove the claim for all $\phi \in P^n(G, X)$ with $\phi(mt) = m^n \phi(t)$. But $\Delta_s \phi \in P^{n-1}(G, X)$ so $\Delta_s \phi(t) = \Delta_s \phi(0)$ for $t \in H_0$. In particular $\Delta_{mt} \phi(t) = \Delta_{mt} \phi(0)$ for $t \in H_0$, $m \in \mathbf{Z}_+$. Thus $\phi((m+1)t) - \phi(t) = \phi(mt) - \phi(0)$, that is $(m+1)^n \phi(t) - \phi(t) = m^n \phi(t)$. Hence $((m+1)^n - (m^n + 1))\phi(t) = 0$. But $n > 1$, so $\phi(t) = 0$ as required. \square

Theorem 2.7. With H_0 as above:

- (a) $P^n(G/H_0, X) = P^n(G, X)$ for $n \in \mathbf{Z}_+$.
- (b) $G/H_0 = \mathbf{R}^m \times F$ for some $m \in \mathbf{Z}_+$ and some torsion free discrete group F .
- (c) If $P^1(G)$ is finite dimensional, then $G/H_0 = \mathbf{R}^m \times \mathbf{Z}^k$ for some $k, m \in \mathbf{Z}_+$ and $P^n(G) = P_R^n(G)$ for all $n \in \mathbf{Z}_+$.

Proof. (a) Let $\pi : G \rightarrow G/H_0$ be the canonical projection. By Proposition 2.6 the map $P^n(G/H_0, X) \rightarrow P^n(G, X)$ defined by $\phi \rightarrow \phi \circ \pi$ is an isomorphism.

(b) Let $\xi = \pi(t) \in G/H_0$, $t \notin H_0$. By definition of H_0 there is a homomorphism $\alpha : G \rightarrow \mathbf{R}$ with $\alpha(t) \neq 0$. Its preimage under the mapping in (a) with $X = \mathbf{C}$ is a homomorphism $a : G/H_0 \rightarrow \mathbf{R}$ with $a(\xi) \neq 0$. Thus by Theorem 2.5, $G/H_0 = \mathbf{R}^m \times F$ for some $m \in \mathbf{Z}_+$ and some torsion free discrete group F .

(c) Let $F_0 \subset F$ be any finitely generated subgroup of F . By [6, Theorem 9.3, p. 90] $F_0 = \mathbf{Z}^k$ for some $k \in \mathbf{Z}_+$. The coordinate functions $\alpha_j : F_0 \rightarrow \mathbf{R}$ extend by [12, Theorem (Kaplansky), p. 44] to homomorphisms $\alpha_j : F \rightarrow \mathbf{R}$. By part (b) they extend further to linearly independent homomorphisms $\alpha_j : G/H_0 \rightarrow \mathbf{R}$. Since $P^1(G)$ is finite dimensional we conclude that there is a maximal such $k \in \mathbf{Z}_+$. Hence $F = \mathbf{Z}^k$ and $P^n(G) = P^n(G/H_0) = P_R^n(G/H_0) = P_R^n(G)$. \square

Corollary 2.8. If $G = J - J$ where $J^\circ \neq \emptyset$ and $P^1(J)$ is finite dimensional, then $G/H_0 = \mathbf{R}^m \times \mathbf{Z}^k$ for some $m, k \in \mathbf{Z}_+$ and $P^n(J)$ is finite dimensional for each $n \in \mathbf{Z}_+$.

Proof. By Theorem 1.5, $P^1(G)$ is finite dimensional. Therefore the result follows from Theorem 2.7 (c). \square

3. A class of groups with $P^n(G)$ finite dimensional

In this section we assume also that X is locally convex so that the Hahn-Banach theorem may be applied. We exhibit a general class of groups G under which $P^n(G)$ is finite dimensional.

Lemma 3.1. (a) If $P^n(J)$ is finite dimensional then $P^n(J, X) = P^n(J) \otimes X$.

(b) If $P_w^n(G)$ is finite dimensional then $P_w^n(G, X) = P_w^n(G) \otimes X$.

Proof. (a) Clearly, $P^n(J) \otimes X \subseteq P^n(J, X)$. For the converse, which is clearly true when $n = 0$, we use induction on n . Let $\{p_1, \dots, p_k\}$ be a basis of a complement Q of $P^{n-1}(J)$ in $P^n(J)$. Since $\deg(p_1) = n$ we can choose $t_1 \in J$ such that $\Delta_{t_1}^n p_1 \neq 0$. Since each $\Delta_{t_1}^n p_j$ is a constant we can set $q_1 = p_1 / \Delta_{t_1}^n p_1$ and choose $\lambda_1 \in \mathbf{C}$ such that $\Delta_{t_1}^n (p_2 - \lambda_1 q_1) = 0$. Set $q_2 = p_2 - \lambda_1 q_1$. Then $\deg(q_2) = n$ for otherwise $q_2 \in Q \cap P^{n-1}(J, \mathbf{C}) = \{0\}$, contradicting the linear independence of p_1, p_2 . Hence we can choose $t_2 \in J$ such that $\Delta_{t_2}^n q_2 \neq 0$. Continuing in this way, we obtain a basis $\{q_1, \dots, q_k\}$ of Q and a subset $\{t_1, \dots, t_k\}$ of J such that $\Delta_{t_i}^n q_j = \delta_{i,j}$. Now let $p \in P^n(J, X)$. For each $x^* \in X^*$ we have $x^* \circ p \in P^n(J)$. Hence $x^* \circ p = \sum_{j=1}^k q_j c_j(x^*) + r(x^*)$ for some $c_j(x^*) \in \mathbf{C}$ and $r(x^*) \in P^{n-1}(J)$. But $x^* \circ \Delta_{t_i}^n p = \Delta_{t_i}^n (x^* \circ p) = c_i(x^*)$ and so $c_i(x^*) = x^*(c_i)$ where $c_i = \Delta_{t_i}^n p$. Moreover, $x^* \circ (p - \sum_{j=1}^k q_j c_j) = r(x^*) \in P^{n-1}(J)$ and so by the Hahn-Banach theorem $\Delta_t^n (p - \sum_{j=1}^k q_j c_j) = 0$ for all $t \in J$ showing $p - \sum_{j=1}^k q_j c_j \in P^{n-1}(J, X)$. By the induction hypothesis $P^{n-1}(J, X) = P^{n-1}(J) \otimes X$ and hence $p \in P^n(J, \mathbf{C}) \otimes X$ as required.

(b) This is proved in the same way as above. \square

Lemma 3.2. Let J_1, J_2 be abelian topological semigroups. If $P^n(J_1)$ is finite dimensional, then

$$P^n(J_1 \times J_2) = \sum_{m=0}^n P^m(J_1) \otimes P^{n-m}(J_2).$$

Proof. The inclusion \supseteq is clear. For the converse, take any $p \in P^n(J_1 \times J_2)$. As in the proof of Lemma 3.1, we can find for each $m = 1, \dots, n$ a basis $\{q_1^m, \dots, q_{k_m}^m\}$ of a complement of $P^{m-1}(J_1)$ in $P^m(J_1)$ and a subset $\{s_1^m, \dots, s_{k_m}^m\}$ of J_1 such that $\Delta_{s_i^m}^m q_j^m = \delta_{i,j}$. Also let $\{q_1^0\}$ be a basis of $P^0(J_1)$. For any $t \in J_2$ we have $p(\cdot, t) \in P^n(J_1)$ and so $p(s, t) = \sum_{m=1}^n \sum_{j=1}^{k_m} q_j^m(s) r_j^m(t)$ for some $r_j^m(t) \in \mathbf{C}$. We prove by backward induction on h that $r_j^h \in P^{n-h}(J_2)$. Now

$$\Delta_{(s_i^n, 0)}^n p(s, t) = \sum_{m=1}^n \sum_{j=1}^{k_m} \Delta_{s_i^n}^n q_j^m(s) r_j^m(t) = r_i^n(t)$$

so each r_i^n is a constant as required. So suppose each $r_j^m \in P^{n-m}(J_2)$ for $n \geq m \geq h+1$ and $1 \leq j \leq k_m$. Then

$$p(s, t) - \sum_{m=h+1}^n \sum_{j=1}^{k_m} q_j^m(s) r_j^m(t) = \sum_{m=0}^h \sum_{j=1}^{k_m} q_j^m(s) r_j^m(t) \in P^n(J_1 \times J_2, \mathbf{C}) \text{ and } \Delta_{(s_i^h, 0)}^h \sum_{m=0}^h \sum_{j=1}^{k_m} q_j^m(s) r_j^m(t) = r_i^h(t). \text{ So each } r_i^h \in P^{n-h}(J_2)$$

and the proposition is proved. \square

Lemma 3.3. Let H be a closed subgroup of G such that G/H is a torsion group.

(a) The restriction map $r : P^n(G, X) \rightarrow P^n(H, X)$ is one-to-one and so $\dim(P^n(G, X)) \leq \dim(P^n(H, X))$.

(b) If also G/H is of bounded order and $P^n(H)$ is finite dimensional, then

$r : P^n(G, X) \rightarrow P^n(H, X)$ is a linear isomorphism.

Proof. (a) Let $p \in P^n(G, X)$ satisfy $p(t) = 0$ for all $t \in H$. Let $s \notin H$ and let $\pi : G \rightarrow G/H$ be the quotient map. Since G/H is a torsion group, $\pi(ks) = 0$, meaning $ks \in H$, for some $k \in \mathbf{N}$. Hence $p(mks) = 0$ for all $m \in \mathbf{N}$. But $p(ms)$ is a polynomial in $m \in \mathbf{Z}_+$ and so is zero. In particular, $p(s) = 0$ showing r is one-to-one.

(b) Let $\{p_1, \dots, p_m\}$ be a basis of $P^n(H)$ and choose $k \in \mathbf{N}$ such that $kt \in H$ for all $t \in G$. Define $q_j(t) = p_j(kt)$ and suppose $\sum_{j=1}^m c_j q_j = 0$ on G for some $c_j \in \mathbf{C}$. Then $\sum_{j=1}^m c_j p_j = 0$ on kG . But kG is a closed subgroup of H such that H/kG is a torsion group. By part (a), $\sum_{j=1}^m c_j p_j = 0$ on H . Hence each $c_j = 0$, showing $\{q_1, \dots, q_m\}$ is linearly independent and $\dim(P^n(G)) \geq \dim(P^n(H))$. Therefore $r : P^n(G, X) \rightarrow P^n(H, X)$ is a linear isomorphism, by (a) when $X = \mathbf{C}$ and then by Lemma 3.1 for general X . \square

Reiter, [9, p.142], introduced the notion of *w-uniform continuity* for functions $\phi \in C(G, X)$, namely $\sup_{t \in G} \|\Delta_h \phi(t)/w(t)\| \rightarrow 0$ as $h \rightarrow 0$ in G . The space of all such functions for which ϕ/w is bounded is denoted by $BUC_w(G, X)$. The weight w has *polynomial growth* of order $N \in \mathbf{Z}_+$ if it satisfies conditions (3.1), (3.2) of [1].

Theorem 3.4. Assume G has an open subgroup G_0 such that $G_0 = \mathbf{R}^m \times K$ for some $m \in \mathbf{Z}_+$ and some compact group K , and $G/G_0 = \mathbf{Z}^k \times F$ for some $k \in \mathbf{Z}_+$ and some torsion group F . Then for each $n \in \mathbf{Z}_+$,

(a) $P^n(G)$ is finite dimensional.

Assume also F is of bounded order.

(b) For each $p \in P^n(G, X)$ there exist $p_j \in P^n(G, X)$ and $q_j \in P^n(G)$ with $q_j(0) = 0$ such that $\Delta_h p(t) = \sum_{j=1}^k p_j(t) q_j(h)$ for all $h, t \in G$.

(c) $P_w(G, X) \subseteq BUC_w(G, X)$.

(d) If also w has polynomial growth N , then $P_w(G)$ is finite dimensional.

Proof. (a) Note that G_0 is also closed (see [12, Appendix B5]). Hence G/G_0 is discrete. Since $G/G_0 = \mathbf{Z}^k \times F$ and \mathbf{Z}^k is a free group, G has a subgroup H isomorphic to \mathbf{Z}^k with $H \cap G_0 = \{0\}$. Since G_0 is open, $G_1 = G_0 \times H$ is open too. This implies G_1 is also closed. Moreover, $G/G_1 = F$. By Lemma 3.3 (a) $\dim P^n(G) \leq \dim P^n(G_1)$. By Examples 1.1 (a), Lemma 2.4 and Lemma 3.2 we conclude that $\dim P^n(G)$ is finite.

(b) Since (b) holds when $G = \mathbf{R}^m \times \mathbf{Z}^k$ it holds for G_1 . Since F has bounded order, (b) follows by Lemma 3.3 (b).

(c) If $p \in P_w(G, X)$ choose p_j, q_j as in (b). Since $\|\Delta_h p(t)\| \leq cw(t) \sum_{j=1}^k \|q_j(h)\|$, where $c = \sup_j \sup_{t \in G} \|p_j(t)/w(t)\|$, it follows that $p \in BUC_w(G, X)$, proving (c).

(d) The assumptions imply $P_w(G, X) \subset P^N(G, X)$ and therefore the result follows from (a). \square

Remarks 3.5. (a) If G is compactly generated, then G satisfies all the assumptions of Theorem 3.4. Indeed, by the principal structure theorem for locally compact abelian groups (see [12, Theorem 2.4.1]), each such group G has an open subgroup $G_0 = \mathbf{R}^m \times K$ for some $m \in \mathbf{Z}_+$ and some compact group K . Hence G/G_0 is discrete. Since G is compactly generated so is G/G_0 and therefore G/G_0 is finitely generated. So $G/G_0 = \mathbf{Z}^k \times F$ for some $k \in \mathbf{Z}_+$ and some finite group F (see Theorem 9.3 in [6]).

(b) There are groups G satisfying all the assumptions of Theorem 3.4 which are not compactly generated. For example, take $F = \mathbf{Z}_2^{\mathbf{R}} = \{f : \mathbf{R} \rightarrow \mathbf{Z}_2\}$, where \mathbf{Z}_2 is the group of order 2, endowed with the discrete topology. Then F is a torsion group of bounded order 2 which is not compactly generated.

(c) There are locally compact groups G for which $P^n(G)$ is infinite dimensional for each $n > 0$. See Examples 1.2 (b).

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